A Fractional Calculus Approach to Modeling the Rehabilitation Dynamics of Stock Markets

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Abstract
The dynamics of stock markets often exhibit complex behaviors that traditional models struggle to capture, particularly in recovery phases following periods of volatility. This paper introduces a novel mathematical model based on fractional calculus to describe and predict the rehabilitation dynamics of stock markets. We formulate a fractional differential equation (FDE) model and validate its effectiveness using historical market data. The model's ability to account for long-term memory effects and non-local interactions offers significant advantages over classical approaches.

Keywords: *Fractional calculus, fractional differential equations, stock market dynamics, healing dynamics, financial modeling, Caputo derivative*

1. INTRODUCTION

The inherent complexity of stock market dynamics poses challenges for conventional modeling techniques, especially in understanding recovery processes post-crises. Fractional calculus provides a powerful framework for capturing these complexities by extending traditional differential equations to non-integer orders, thereby incorporating memory-dependent behaviors.
The concept of healing in the stock market refers to the gradual recovery and stabilization of a portfolio after a period of losses. Traditional models often focus on risk management and portfolio

optimization without adequately addressing the psychological and strategic adjustments traders make in response to adverse market conditions. This paper addresses this gap by incorporating the following; liquidity restoration, price momentum recovery, market sentiment recovery, volatility reduction, price recovery and price stabilization which are involved in trading in the stock market. These practices are essential for improving decision-making and avoiding repeated errors.
We now give brief explanations of the instruments of rehabilitation mentioned above.

(a) Liquidity Restoration in Financial Markets

Liquidity restoration refers to the process by which a financial market or an asset regains its ability to trade smoothly after a disruption. Disruptions to liquidity can occur due to various factors such as market crashes, economic crises, regulatory changes, or even significant news events. Restoring liquidity is critical because it ensures that assets can be bought and sold with minimal impact on their prices, which in turn promotes market stability and investor confidence.
Mechanisms of Liquidity Restoration include;

1. **Central Bank Interventions**: Central banks often play a crucial role in restoring liquidity by providing emergency funding to financial institutions or engaging in open market operations. For instance, during the 2008 financial crisis, the Federal Reserve implemented measures such as quantitative easing to inject liquidity into the market (Fawley & Neely, 2013).
2. **Market-Maker Support**: Market makers, who provide buy and sell quotes for assets, can help restore liquidity by increasing their presence in the market during times of stress. Their participation ensures that there are always counterparties available for trades, reducing the bid-ask spread and enhancing market depth (Menkveld, 2013).
3. **Regulatory Measures**: Regulators may implement temporary measures to stabilize markets, such as short-selling bans or circuit breakers. These actions can prevent panic selling and give the market time to stabilize, thereby aiding in the restoration of liquidity (Clarke et al., 2018).
4. **Investor Behavior**: The gradual return of investor confidence can also lead to liquidity restoration. As investors regain trust in the market, they are more likely to engage in trading, which increases liquidity. Behavioral finance suggests that this process can be self-reinforcing as increasing liquidity attracts more participants, further enhancing market stability (Barberis & Thaler, 2003).

Restoring liquidity is not always straightforward. The effectiveness of interventions can depend on various factors, including the underlying cause of the liquidity disruption, the speed of response, and the broader economic environment. Moreover, excessive reliance on external interventions, such as central bank support, can lead to moral hazard, where market participants take on more risk, expecting that they will be bailed out during crises (Farhi &Tirole, 2012).

(b) Price Momentum Recovery in Financial Markets

Price momentum refers to the tendency of an asset's price to continue moving in the same direction—either upward or downward—over a period. Price momentum recovery occurs when an asset that has experienced a decline in momentum, often due to negative news, market corrections, or economic shocks, begins to regain its previous trajectory, leading to a resumption of upward price movement.

Among the factors contributing to price momentum recovery are;

1. **Market Sentiment Shift**: A significant driver of price momentum recovery is a change in market sentiment. When investors regain confidence in an asset's future prospects, often due to positive news or improved economic indicators, they are more likely to buy, pushing the price higher and restoring momentum. Behavioral finance suggests that this shift can be driven by the psychological effects of optimism and herding behavior among investors (Barberis et al., 1998).
2. **Earnings Surprises and Positive News**: Positive earnings reports or other favorable news can trigger a recovery in price momentum. Investors often react strongly to earnings surprises, leading to increased buying pressure and a subsequent rise in prices (Jegadeesh & Titman, 1993). This reaction can reinforce the momentum as more investors pile in, expecting the trend to continue.
3. **Technical Indicators**: Technical analysis often plays a role in identifying momentum recovery. Traders look for patterns such as moving averages, relative strength index (RSI), or MACD (Moving Average Convergence Divergence) to signal that an asset's price momentum is shifting back to an upward trend (Moskowitz, Ooi, & Pedersen, 2012). When these indicators suggest a recovery, it can prompt buying activity that further fuels momentum.
4. **Macroeconomic Conditions**: Broader economic improvements, such as GDP growth, lower unemployment rates, or favorable interest rate environments, can support price momentum recovery. These conditions improve the overall market environment, encouraging investment and driving up asset prices (Asness et al., 2013).

Despite the potential for recovery, price momentum is subject to reversal, especially in the face of adverse economic conditions or market-wide corrections. Additionally, momentum-based strategies can lead to increased volatility and risk, particularly if many investors attempt to exit positions simultaneously during a downturn (Daniel & Moskowitz, 2016).

Moreover, the sustainability of momentum recovery can be influenced by factors such as market liquidity, investor sentiment, and external shocks. For instance, geopolitical events or unexpected economic data releases can abruptly alter momentum, leading to renewed price declines.

(c). Market Sentiment Recovery in Financial Markets

Market sentiment refers to the overall attitude of investors toward a particular financial market or asset, often driven by a combination of economic indicators, news, and psychological factors. Market sentiment recovery occurs when negative or bearish sentiment shifts toward a more positive or bullish outlook. This recovery is critical as it can drive investment decisions, influence asset prices, and restore market stability.

The drivers of market sentiment recovery include;

1. **Positive Economic Data**: Improvements in key economic indicators such as GDP growth, employment rates, or inflation figures can lead to a recovery in market sentiment. When investors see signs of economic stability or growth, their confidence in the market increases, prompting more buying activity and lifting sentiment. For example, a report showing unexpectedly high job creation can shift sentiment from bearish to bullish (Baker & Wurgler, 2007).

2. **Government and Central Bank Interventions**: Policymakers often play a pivotal role in restoring market sentiment during times of distress. Actions such as interest rate cuts, fiscal stimulus, or asset purchase programs by central banks can reassure investors that economic conditions will improve. The announcement of these measures can trigger a shift in sentiment as investors anticipate a more favorable environment for asset prices (Romer & Romer, 2000).
3. **Corporate Earnings and Guidance**: Positive earnings reports and optimistic future guidance from companies can lead to a recovery in sentiment, especially if these reports exceed market expectations. When major firms report strong financial performance or announce strategic initiatives that promise future growth, it can signal to investors that the broader economy is on the mend, leading to improved sentiment (Tetlock, 2007).
4. **Market Stability and Reduced Volatility**: As market volatility decreases and stability returns, investor confidence often improves. Volatility indices, such as the VIX, often serve as a barometer of sentiment, and a decline in these indices typically corresponds with a recovery in sentiment. Reduced volatility suggests that the market is less likely to experience large swings, which can encourage investors to re-enter the market (Whaley, 2000).
5. **Media Influence and Investor Psychology**: Media coverage and public narratives can significantly impact market sentiment. Positive news stories, expert opinions, and optimistic forecasts can shift investor psychology from fear to optimism, leading to a recovery in sentiment. Behavioral finance research highlights how narratives and collective psychology can drive market trends, making media a powerful tool in sentiment recovery (Shiller, 2017).

While market sentiment recovery is essential for the resumption of normal trading conditions, it can be fragile. External shocks, such as geopolitical events or unexpected economic downturns, can quickly reverse positive sentiment. Additionally, sentiment recovery often precedes tangible economic recovery, leading to potential overvaluation of assets and subsequent corrections if the economic improvement does not materialize as expected (Baker & Wurgler, 2006).

(d) Volatility Reduction in Financial Markets

Volatility refers to the degree of variation in the price of a financial asset over time. High volatility typically indicates higher risk, as prices can fluctuate dramatically in short periods. Volatility reduction is a critical aspect of stabilizing financial markets, as it fosters investor confidence, encourages long-term investments, and supports economic growth.

Some of the mechanisms for volatility reduction are;

1. **Central Bank Interventions**: Central banks play a significant role in reducing market volatility through monetary policy tools. By adjusting interest rates, conducting open market operations, or implementing quantitative easing, central banks can influence the liquidity and overall risk environment in the markets. For example, during the 2008 financial crisis, the Federal Reserve's interventions helped to calm markets and reduce volatility (Bekaert, Hoerova, & Duca, 2013).
2. **Regulatory Measures**: Regulatory bodies can implement measures designed to stabilize markets and reduce volatility. These include circuit breakers, which halt trading during extreme price movements, and short-selling restrictions that prevent excessive downward

pressure on asset prices. Such interventions are designed to prevent panic selling and give the market time to stabilize, ultimately reducing volatility (Harris, 1998).

3. **Derivatives and Hedging Strategies**: The use of derivatives, such as options and futures, allows investors to hedge against potential losses, thereby reducing their exposure to risk and contributing to overall market stability. By managing risk effectively, these instruments can dampen the impact of sudden price changes, leading to lower volatility (Black, 1976).
4. **Market Liquidity**: High levels of market liquidity contribute to volatility reduction by ensuring that there are enough buyers and sellers to absorb large orders without causing significant price swings. Market makers and high-frequency traders often provide liquidity, helping to smooth out price fluctuations and reduce volatility (Menkveld, 2013).
5. **Investor Sentiment and Behavioral Factors**: Investor sentiment plays a crucial role in market volatility. When sentiment is positive, markets tend to be more stable, as investors are less likely to engage in panic selling. Behavioral finance research suggests that promoting long-term thinking and discouraging herd behavior can contribute to reduced volatility (Shiller, 1981).

While reducing volatility is generally beneficial, there are challenges and trade-offs. Over-reliance on regulatory or central bank interventions can lead to market distortions, where prices do not reflect true underlying risks. Additionally, reducing volatility too much can encourage excessive risk-taking, as investors may become complacent, believing that markets will remain stable indefinitely (Danielsson et al., 2012).

Furthermore, global events such as geopolitical tensions or economic crises can trigger sudden spikes in volatility, making it difficult for traditional mechanisms to maintain stability. This highlights the importance of a multi-faceted approach to volatility management, combining policy interventions, market infrastructure improvements, and investor education.

(e) Price Recovery in Financial Markets

Price recovery refers to the process by which the price of a financial asset rebounds after experiencing a significant decline. This phenomenon is essential for the stabilization of markets following downturns or crashes and can be driven by a combination of economic, psychological, and technical factors.

Among the factors contributing to price recovery are;

1. **Improvement in Economic Fundamentals**: A primary driver of price recovery is the improvement in underlying economic conditions. When economic indicators such as GDP growth, employment rates, or corporate earnings start showing positive trends, investors regain confidence, leading to increased demand for assets and, consequently, a recovery in prices. For instance, the global stock market recovery following the 2008 financial crisis was largely driven by improving economic conditions and corporate earnings (Reinhart & Rogoff, 2009).
2. **Government and Central Bank Interventions**: Policy interventions can play a critical role in facilitating price recovery. Governments may introduce fiscal stimulus packages, while central banks might cut interest rates or engage in quantitative easing to support economic activity. These actions can boost investor confidence, leading to a rebound in asset prices. The swift recovery of asset prices during the COVID-19 pandemic, for

example, was partly attributed to the massive fiscal and monetary interventions by governments and central banks globally (Gopinath, 2020).

3. **Investor Sentiment and Market Psychology**: Investor sentiment often drives price recovery. Positive news, such as vaccine developments during a pandemic or breakthroughs in trade negotiations, can lead to a shift in market psychology from fear to optimism. This shift can create a self-reinforcing cycle where rising prices attract more investors, further driving the recovery (Shiller, 2003).
4. **Technical Factors and Market Mechanics**: Technical analysis and market mechanics also contribute to price recovery. When asset prices fall to certain support levels or when oversold conditions are identified, technical traders may begin buying, anticipating a reversal. Additionally, short-sellers may cover their positions as prices stabilize, adding further upward pressure and contributing to the recovery (Lo &MacKinlay, 1990).
5. **Sector Rotation and Reallocation**: During periods of recovery, investors often rotate into sectors that are expected to perform well in the new economic environment. This reallocation of capital can lead to price recovery in specific sectors or asset classes, particularly those perceived as undervalued or likely to benefit from emerging economic trends (Fama & French, 1993).

Price recovery is not always guaranteed and can face significant challenges. If economic conditions fail to improve or if new negative shocks occur, price recovery may be stalled or reversed. Additionally, premature recovery attempts can lead to "false dawns," where prices rise temporarily before falling again, leading to increased volatility and investor uncertainty (Kindleberger &Aliber, 2011).

Moreover, price recovery can be uneven across different markets and asset classes, with some sectors recovering faster than others. This can create disparities and affect overall market sentiment.

(f) Price Stabilization in Financial Markets

Price stabilization refers to efforts to reduce excessive volatility and maintain relatively consistent asset prices in financial markets. Stabilized prices promote market confidence, facilitate investment, and contribute to overall economic stability. Various mechanisms and strategies are employed by market participants, governments, and regulatory bodies to achieve price stabilization.

The following contribute to mechanisms for price stabilization**;**

1. **Central Bank Interventions**: Central banks play a crucial role in price stabilization by influencing interest rates, controlling inflation, and providing liquidity to financial markets. For example, by adjusting the federal funds rate, the Federal Reserve can influence borrowing costs, which in turn affects consumer spending, business investment, and overall economic activity. These actions help to stabilize asset prices by preventing excessive inflation or deflation (Bernanke & Gertler, 2001).
2. **Government Policies and Regulation**: Governments and regulatory bodies can implement policies that directly or indirectly stabilize prices. For instance, during financial crises, governments may introduce fiscal stimulus packages to support economic activity or impose regulations such as price floors or ceilings to prevent extreme price movements in critical markets (Stiglitz, 2010). Additionally, the introduction of circuit breakers in

stock exchanges is a regulatory measure designed to temporarily halt trading during sharp price declines, giving the market time to stabilize (Harris, 1998).

3. **Market Makers and Liquidity Providers**: Market makers and liquidity providers contribute to price stabilization by continuously offering to buy and sell securities, thereby ensuring that there is always a counterparty for transactions. This reduces the likelihood of large price swings caused by temporary imbalances between supply and demand. By maintaining tight bid-ask spreads, market makers help to stabilize prices and reduce volatility (Madhavan & Smidt, 1993).
4. **Hedging and Derivative Instruments**: The use of derivatives, such as options and futures, allows investors to hedge against price fluctuations, which can contribute to overall market stability. By locking in prices or protecting against adverse price movements, these financial instruments help to smooth out the impact of market volatility, thereby contributing to price stabilization (Black & Scholes, 1973).
5. **Investor Sentiment and Behavioral Factors**: Investor behavior plays a significant role in price stabilization. Positive investor sentiment, driven by confidence in the economy or corporate earnings, can lead to more stable pricing. Conversely, irrational behavior or panic selling can lead to price instability. Behavioral finance emphasizes the importance of managing investor expectations and promoting rational decision-making to achieve price stability (Shiller, 1981).

Despite efforts to stabilize prices, several challenges persist. External shocks such as geopolitical events, natural disasters, or sudden economic downturns can disrupt markets and lead to price instability. Additionally, overly aggressive stabilization efforts, such as excessive government intervention, can lead to market distortions, moral hazard, and long-term inefficiencies (Blinder, 2010).

Moreover, globalization and the interconnectedness of financial markets mean that instability in one market can quickly spread to others, complicating efforts to stabilize prices on a global scale.

Fractional calculus as a generalization of classical calculus, has proven to be a powerful tool in modeling memory and hereditary properties in various systems. Its application in financial modeling allows for the incorporation of past events into current market dynamics, providing a more realistic and nuanced understanding of market behavior. By employing fractional calculus, this paper introduces a model that captures the memory effect inherent in trading strategies and market movements, offering a deeper insight into the recovery process.

Fractional calculus has gained traction in various scientific disciplines for its ability to model long-term dependencies and irregular behaviors more accurately than integer-order calculus (Podlubny, 1999). In finance, fractional models have been successfully applied to model volatility, price dynamics, and risk management (Mainardi, 2010; Zeng et al., 2012).

II MATHEMATICAL FORMULATION

In this section we give the derivation of the mathematical model.The mathematical model for rehabilitating the stock market will comprise of liquidity restoration, price momentum recovery, market sentiment recovery, volatility reduction, price recovery and price stabilization made of six partial differential equations.

The model is given as follows

$$\frac{du}{dt} = \Omega h + \varphi p + db + \tau a + \eta c + v r - \xi u \quad (1)$$

where $h, p, b, a, c,$ and r, liquidity restoration, price momentum recovery, market sentiment recovery, volatility reduction, price recovery and price stabilization respectively, and $\Omega, \varphi, d, \tau, \eta, v$ are the rates at which liquidity restoration, price momentum recovery, market sentiment recovery, volatility reduction, price recovery and price stabilization are introduced in healing the stock market respectively. The ξu term represents the factors that cause stock market destabilization and the factors that inhibit the healing of the stock market.
We proceed to derive the model considering the components for healing as follows:

2.1. Liquidity restoration $h(x, t)$

$$\frac{\partial h}{\partial t} = D_h \frac{\partial^2 h}{\partial x^2} + \mu b - \beta h \quad (2)$$

where:

- $h(x, t)$represents the healing factor, which could be a variable related to market recovery over time and space (e.g., price recovery, sentiment index, or volatility reduction).
- $\frac{\partial h}{\partial t}$ is the time derivative of h, indicating how the healing factor changes over time.
- $D_h \frac{\partial^2 h}{\partial x^2}$is the diffusion term, where $D_h$is the diffusion coefficient. This term models the spreading or smoothing of the healing effect across the market.
- $\mu b$is a source term, where $\mu$is a constant and $b(x, t)$ represents an external influence or market intervention that contributes positively to the healing factor.
- βh is a reaction or decay term, where β is a constant that models the rate at which the healing factor decays over time.

This is a partial differential equation (PDE) commonly used to model processes involving diffusion, reaction, and growth. This healing factor represents the restoration of market liquidity, with μbrepresenting efforts by central banks or market makers to inject liquidity into the market.

2.2.Price Momentum Recovery$\rho(x, t)$

This stock market healing factor can be modeled as

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} (pA) \frac{\partial h}{\partial x} + \gamma hb - \zeta p \quad (3)$$

where:

- $\rho(x, t)$ represents the healing factor, which might be linked to the recovery of prices, liquidity, or market sentiment.
- $\frac{\partial \rho}{\partial t}$ is the time derivative of ρ indicating the rate of change of the healing factor over time.
- $p(x, t)$represents another market variable, such as price, sentiment, or liquidity.
- $h(x, t)$ is a function that describes a different aspect of the market, such as a measure of the rate of recovery.
- $A(x, t)$is a coefficient that could represent the influence of factors like market volatility or trading volume on the healing process.
- $\gamma$and ζ are constants describing the strength of specific effects on the healing factor.

- $b(x,t)$is an external influence, such as a government intervention or market manipulation.

Each of these terms contributes to a dynamic interplay between market variables, influencing how the overall healing process unfolds over time and across different market segments.
We put

$$V = A\frac{\partial h}{\partial x} \tag{4}$$

where A is constant that is positive where otherwise the stock market healing process is not taking place.

2.3. **Market Sentiment Recovery$\boldsymbol{b}(\boldsymbol{x},\boldsymbol{t})$**
This stock market healing factor can be modeled as

$$\frac{\partial b}{\partial t} = Ap\ \frac{\partial h}{\partial x} + \varepsilon b(k-b) \tag{5}$$

where:

- $b(x,t)$represents the healing factor, which might be related to market sentiment, liquidity, or another aspect of market recovery.
- $\frac{\partial b}{\partial t}$ is the time derivative of b, indicating how this healing factor changes over time.
- A is a constant that modulates the influence of the price $p(x,t)$ on the healing factor.
- $p(x,t)$could represent the stock price or a related market variable.
- $h(x,t)$might represent another aspect of market recovery, such as sentiment or volatility.
- $\epsilon$is a constant that determines the strength of the logistic growth or decay term.
- k is a carrying capacity or maximum value that $b(x,t)$can approach.

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2.4. Volatility Reduction$\boldsymbol{a}(\boldsymbol{x},\boldsymbol{t})$
This healing factor can be modeled as

$$\frac{\partial a}{\partial t} = Ap\frac{\partial^2 h}{\partial x^2}\ +\ \aleph a(k-a) \tag{6}$$

where:

- $a(x,t)$represents the healing factor, which could relate to price recovery, market sentiment, liquidity, or another aspect of market stabilization.
- $\frac{\partial a}{\partial t}$ is the time derivative of a, representing how the healing factor changes over time.
- A is a constant that modulates the influence of the market variable $p(x,t)$.
- $p(x,t)$represents the stock price, trading volume, or another relevant market metric.
- $h(x,t)$represents a related market variable, such as sentiment, volatility, or another factor influencing the healing process.
- $\frac{\partial^2 h}{\partial x^2}$ is the second spatial derivative of $h(x,t)$, which describes how the curvature of $h(x,t)$ influences a. This term often appears in diffusion processes.
- $\aleph$ is a constant representing the rate of logistic growth or decay.
- k is the carrying capacity or maximum value that $a(x,t)$can approach.
-

2.5. Price Recovery$\boldsymbol{C}(\boldsymbol{x},\boldsymbol{t})$
Price Recoverystock market healing factor can be modeled as

$$\frac{\partial C}{\partial t} = D_h \frac{\partial^2 h}{\partial x^2} + yb - gc \quad (7)$$

where:

- $C(x,t)$)represents the healing factor, which could be related to price recovery, market sentiment, liquidity, or another aspect of market stabilization.
- $\frac{\partial C}{\partial t}$is the time derivative of C, indicating how this healing factor changes over time.
- D_h is the diffusion coefficient, which describes how the healing factor spreads across the market.
- $\frac{\partial^2 h}{\partial x^2}$is the second spatial derivative of $h(x,t)$, representing the curvature or spatial variation of $h(x,t)$.
- y is a constant that modulates the influence of $b(x,t)$, which could represent an external influence or another market variable, such as liquidity injection, investor sentiment, or trading volume.
- g is a constant representing the rate at which $C(x,t)$decays or diminishes over time, which could be due to market forces, investor behavior, or regulatory impacts.

2.6. Price Stabilization$\boldsymbol{r}(\boldsymbol{x},\boldsymbol{t})$

Price Stabilization as stock market healing factor can be modeled as

$$\frac{\partial r}{\partial t} = D_h \frac{\partial^2 h}{\partial x^2} + \sigma b - \theta r \quad (8)$$

where:

- r(x, t)represents the healing factor, which might be related to market recovery, such as price stabilization, reduction in volatility, or restoration of liquidity.
- $\frac{\partial r}{\partial t}$is the time derivative of $r(x,t)$, indicating how this healing factor evolves over time.
- D_h is the diffusion coefficient, which controls the rate at which the healing factor spreads across the market.
- $\frac{\partial^2 h}{\partial x^2}$ is the second spatial derivative of $h(x,t)$, representing the spatial curvature or variation of $h(x,t)$which could be a related market variable, such as investor sentiment or market volatility.
- $\sigma$is a constant that determines the strength of the external influence $b(x,t)$on the healing factor.
- $\theta$is a constant that represents the rate at which the healing factor $r(x,t)$ decays over time.

2.7Fractional Diffusion Model for Stock Market Rehabilitation

Now consider the fractional differential model for stock market rehabilitation using the first-order Caputo derivatives of order α given by the following non-linear system

$$D_t^{\alpha_1} h(t) = D_h \frac{\partial^2 h}{\partial x^2} + \mu b - \beta h \quad (9)$$

$$D_t^{\alpha_2} \rho(t) = \frac{\partial}{\partial x}(pA)\frac{\partial h}{\partial x} + \gamma hb - \zeta p \quad (10)$$

$$D_t^{\alpha_3} b(t) = Ap\frac{\partial h}{\partial x} + \varepsilon b(k-b) \quad (11)$$
$$D_t^{\alpha_4} a(t) = Ap\frac{\partial^2 h}{\partial x^2} + \aleph a(k-a) \quad (12)$$
$$D_t^{\alpha_5} c(t) = D_h\frac{\partial^2 h}{\partial x^2} + yb - gc \quad (13)$$
$$D_t^{\alpha_6} r(t) = D_h\frac{\partial^2 h}{\partial x^2} + \sigma b - \theta r \quad (14)$$

with the initial/boundary conditions given as

$$\left.\frac{\partial h}{\partial x}\right|_{x=0} = \left.\frac{\partial h}{\partial x}\right|_{x=L} = 0 \qquad p(0,t) \quad (15)$$
$$h(x,0) = \begin{cases} \frac{k_2 b,}{k_4}, 0 < x < w \\ 0, w < x \leq L \end{cases} \quad (16)$$
$$b(x,0) = \begin{cases} b, & 0 < x < w \\ 0, & w < x \leq L \end{cases} \quad (17)$$
$$a(x,0) = \begin{cases} a, & 0 < x < w \\ 0, & w < x \leq L \end{cases} \quad (18)$$
$$c(x,0) = \begin{cases} c, & 0 < x < w \\ 0, & w < x \leq L \end{cases} \quad (19)$$
$$p(x,0) = 0, 0 \leq x \leq Lb(x,0), 0 \leq x \leq L \quad (20)$$

III THE NUMERICAL SOLUTION

3.1 Numerical Solution using Laplace Adomian Decomposition Method

The Adomian Decomposition Method (ADM) is an analytical technique designed to solve a broad spectrum of linear and nonlinear differential equations, integral equations, and algebraic equations. Introduced by George Adomian in the 1980s, the ADM decomposes the solution of an equation into a series of functions, typically represented as an infinite series. The method is particularly useful for nonlinear problems, as it utilizes Adomian polynomials to systematically break down and simplify nonlinear operators without requiring linearization or discretization. This characteristic makes ADM a versatile tool for modeling complex systems across various fields, including engineering and finance (Adomian, 1994).

An extension of this method, the Laplace Adomian Decomposition Method (LADM), integrates the Laplace transform into the decomposition process to efficiently handle initial conditions, particularly in boundary value and initial value problems. By transforming the differential equation into an algebraic equation in the Laplace domain, the ADM can be applied more effectively, and the inverse Laplace transform is used to revert the solution back to the time domain. This combination enhances the ADM's ability to solve complex initial value problems while retaining its core advantages, such as avoiding perturbation and offering a rapidly converging series solution (Wazwaz, 2009).

Consider the fractional order of our model.

$D_t^{\alpha_1}h(t) = D_h\frac{\partial^2 h}{\partial x^2} + \mu b - \beta h$ (21)

$D_t^{\alpha_2}\rho(t) = \frac{\partial}{\partial x}(pA)\frac{\partial h}{\partial x} + \gamma hb - \zeta p$ (22)

$D_t^{\alpha_3}b(t) = Ap\frac{\partial h}{\partial x} + \varepsilon b(k-b)$ (23)

$D_t^{\alpha_4}a(t) = Ap\frac{\partial^2 h}{\partial x^2} + \aleph a(k-a)$ (24)

$D_t^{\alpha_5}c(t) = D_h\frac{\partial^2 h}{\partial x^2} + yb - gc$ (25)

$D_t^{\alpha_6}r(t) = D_h\frac{\partial^2 h}{\partial x^2} + \sigma b - \theta r$ (26)

The Laplace transform of equations (21) to (26) are

$\mathcal{L}[D_t^{\alpha_1}h(t)] = \mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \mu b - \beta h]$ (27)

$\mathcal{L}[D_t^{\alpha_2}\rho(t)] = \mathcal{L}[\frac{\partial}{\partial x}(pA)\frac{\partial h}{\partial x} + \gamma hb - \zeta p]$ (28)

$\mathcal{L}[D_t^{\alpha_3}b(t)] = \mathcal{L}[Ap\frac{\partial h}{\partial x} + \varepsilon b(k-b)]$ (29)

$\mathcal{L}[D_t^{\alpha_4}a(t)] = \mathcal{L}[Ap\frac{\partial^2 h}{\partial x^2} + \aleph a(k-a)]$ (30)

$\mathcal{L}[D_t^{\alpha_5}c(t)] = \mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + yb - gc]$ (31)

$\mathcal{L}[D_t^{\alpha_6}r(t)] = \mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \sigma b - \theta r]$ (32)

The application of the definition of Laplace transform of Caputo derivative to the left hand sides of equations (27) to (32) give;

$s^{\alpha_1}\mathcal{L}[h(t)] - s^{\alpha_1 - 1}h(0) = \mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \mu b - \beta h]$ (33)

$s^{\alpha_2}\mathcal{L}[p(t)] - s^{\alpha_2 - 1}p(0) = \mathcal{L}[\frac{\partial}{\partial x}(pA)\frac{\partial h}{\partial x} + \gamma hb - \zeta p]$ (34)

$s^{\alpha_3}\mathcal{L}[b(t)] - s^{\alpha_3 - 1}b(0) = \mathcal{L}[Ap\frac{\partial h}{\partial x} + \varepsilon b(k-b)]$ (35)

$s^{\alpha_4}\mathcal{L}[a(t)] - s^{\alpha_4 - 1}a(0) = \mathcal{L}[Ap\frac{\partial^2 h}{\partial x^2} + \aleph a(k-a)]$ (36)

$s^{\alpha_5}\mathcal{L}[c(t)] - s^{\alpha_5 - 1}c(0) = \mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + yb - gc]$ (37)

$s^{\alpha_6}\mathcal{L}[r(t)] - s^{\alpha_6 - 1}r(0) = \mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \sigma b - \theta r]$ (38)

The division of equations (33), (34), (35), (36), (37), and (38) bys^{α_1}, s^{α_2} , $s^{\alpha_3}s^{\alpha_4}$, $s^{\alpha_5}$and ,s^{α_6} respectively give equations (39) to (44)

$\mathcal{L}[h(t)] - s^{-1}h(0) = \frac{1}{s^{\alpha_1}}\mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \mu b - \beta h]$ (39)

$\mathcal{L}[p(t)] - s^{-1}p(0) = \frac{1}{s^{\alpha_2}}\mathcal{L}[\frac{\partial}{\partial x}(pA)\frac{\partial h}{\partial x} + \gamma hb - \zeta p]$ (40)

$\mathcal{L}[b(t)] - s^{-1}b(0) = \frac{1}{s^{\alpha_3}}\mathcal{L}[Ap\frac{\partial h}{\partial x} + \varepsilon b(k-b)]$ (41)

$\mathcal{L}[a(t)] - s^{-1}a(0) = \frac{1}{s^{\alpha_4}}\mathcal{L}[Ap\frac{\partial^2 h}{\partial x^2} + \aleph a(k-a)]$ (42)

$\mathcal{L}[c(t)] - s^{-1}c(0) = \frac{1}{s^{\alpha_5}}\mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + yb - gc]$ (43)

$\mathcal{L}[r(t)] - s^{-1}r(0) = \frac{1}{s^{\alpha_6}}\mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \sigma b - \theta r]$ (44)

Further, solving equations (39) to (44) yield equations (45) to(50)

$$\mathcal{L}[h(t)] = s^{-1}h(0) + \frac{1}{s^{\alpha_1}}\mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \mu b - \beta h] \quad (45)$$

$$\mathcal{L}[p(t)]) = s^{-1}p(0 + \frac{1}{s^{\alpha_2}}\mathcal{L}[\frac{\partial}{\partial x}(pA)\frac{\partial h}{\partial x} + \gamma hb - \zeta p] \quad (46)$$

$$\mathcal{L}[b(t)] = s^{-1}b(0) + \frac{1}{s^{\alpha_3}}\mathcal{L}[Ap\frac{\partial h}{\partial x} + \varepsilon b(k-b)] \quad (47)$$

$$\mathcal{L}[a(t)] = s^{-1}a(0) + \frac{1}{s^{\alpha_4}}\mathcal{L}[Ap\frac{\partial^2 h}{\partial x^2} + \aleph a(k-a)] \quad (48)$$

$$\mathcal{L}[c(t)] = s^{-1}c(0) + \frac{1}{s^{\alpha_5}}\mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + yb - gc] \quad (49)$$

$$\mathcal{L}[r(t)] = s^{-1}r(0) + \frac{1}{s^{\alpha_6}}\mathcal{L}[D_h\frac{\partial^2 h}{\partial x^2} + \sigma b - \theta r] \quad (50)$$

The inverse Laplace transforms of equations (45) through (50) give equations (51) to (56);

$$[h_n(t)] = \mathcal{L}^{-1}\left[\frac{h(0)}{s}\right] + \mathcal{L}^{-1}[\frac{1}{s^{\alpha_1}}\mathcal{L}[D_h\frac{\partial^2 h_n}{\partial x^2} + \mu b_n - \beta h_n]] \quad (51)$$

$$[p_n(t)] = \mathcal{L}^{-1}\left[\frac{p(0)}{s}\right] + \mathcal{L}^{-1}[\frac{1}{s^{\alpha_2}}\mathcal{L}[\frac{\partial}{\partial x}(pA)\frac{\partial h_n}{\partial x} + \gamma h_n b_n - \zeta p_n]] \quad (52)$$

$$[b_n(t)] = \mathcal{L}^{-1}\left[\frac{b(0)}{s}\right] + \mathcal{L}^{-1}[\frac{1}{s^{\alpha_3}}\mathcal{L}[Ap\frac{\partial h_n}{\partial x} + \varepsilon b_n(k-b_n)]] \quad (53)$$

$$[a_n(t)] = \mathcal{L}^{-1}\left[\frac{a(0)}{s}\right] + \mathcal{L}^{-1}[\frac{1}{s^{\alpha_4}}\mathcal{L}[Ap\frac{\partial^2 h_n}{\partial x^2} + \aleph a_n(k-a_n)]] \quad (54)$$

$$[c_n(t)] = \mathcal{L}^{-1}\left[\frac{c(0)}{s}\right] + \mathcal{L}^{-1}[\frac{1}{s^{\alpha_5}}\mathcal{L}[D_h\frac{\partial^2 h_n}{\partial x^2} + yb_n - gc_n]] \quad (55)$$

$$[r_n(t)] = \mathcal{L}^{-1}\left[\frac{r(0)}{s}\right] + \mathcal{L}^{-1}[\frac{1}{s^{\alpha_6}}\mathcal{L}[D_h\frac{\partial^2 h_n}{\partial x^2} + \sigma b_n - \theta r_n]] \quad (56)$$

Putting the solutions of h(t), p(t), b(t), a(t), c(t), and r(t) as an infinite series solution, we have

$$h(t) = \sum_{n=0}^{\infty} h(t)\ ;\ p(t) = \sum_{n=0}^{\infty} p(t)\ ;\ b(t) = \sum_{n=0}^{\infty} b(t) \quad (57)$$

$$a(t) = \sum_{n=0}^{\infty} a(t)\ ;\ c(t) = \sum_{n=0}^{\infty} c(t)\ ;\ r(t) = \sum_{n=0}^{\infty} r(t) \quad (58)$$

From equations (51), through (56), we have

$$h_0 = \mathcal{L}^{-1}\left\{\frac{h(0)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{k_2/k_4}{s}\right\} = \frac{k_2 b}{k_4} \quad (59)$$

$$p_o = \mathcal{L}^{-1}\left\{\frac{\rho(0)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{0}{s}\right\} = 0 \quad (60)$$

$$b_o = \mathcal{L}^{-1}\left\{\frac{b(0)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{b}{s}\right\} = b \quad (61)$$

$$[a_o(t)] = \mathcal{L}^{-1}\left[\frac{a(0)}{s}\right] = \mathcal{L}^{-1}\left[\frac{a}{s}\right] = a \quad (62)$$

$$[c_o(t)] = \mathcal{L}^{-1}\left[\frac{c(0)}{s}\right] = \mathcal{L}^{-1}\left[\frac{0}{s}\right] = 0 \quad (63)$$

$$[r_o(t)] = \mathcal{L}^{-1}\left[\frac{r(0)}{s}\right] = \mathcal{L}^{-1}\left[\frac{0}{s}\right] = 0 \quad (64)$$

From equations (59)to (64)we get

$$h_0 = \frac{k_2 b}{k_4} \quad , p_o = c_o = r_o = 0, b_o = ba_o = a \quad (65)$$

To obtain the values of $h_1, p_1, b_{1,}, a_1, c_1,$ andr_1, we decompose the non-linear terms hb and b^2 involved in the model by Adomian polynomial as follows

$$hb = \sum_{i=0}^{\infty} M_n; \quad b^2 = \sum_{i=0}^{\infty} J_n \quad (66)$$

where $M_n$andJ_n are Adomian polynomials given by

$$M_n = \frac{1}{\Gamma(n+1)}\frac{d}{d\lambda^n}[\sum_{k=0}^{n}\lambda^k h_k \sum_{k=0}^{n}\lambda^k b_k]|_{\lambda=0} \quad (67)$$

$$J_n = \frac{1}{\Gamma(n+1)} \frac{d^n}{d\lambda^n} [\sum_{k=0}^{n} \lambda^k h_k \sum_{k=0}^{n} \lambda^k h_k]|_{\lambda=0} \tag{68}$$

From equation (51), we have

$$h_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha_1}} \mathcal{L}[D_h \frac{\partial^2 h_0}{\partial x^2} + \mu b_0 - \beta h_0]\right\} \tag{69}$$

$$h_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\propto_1}} \mathcal{L}[D_h \frac{\partial^2 k_2 b/k_4}{\partial x^2} + \mu b - \beta \frac{k_2 b}{k_4}]\right\} \tag{70}$$

$$D_h \frac{\partial^2 k_2 b/k_4}{\partial x^2} = 0 \tag{71}$$

$$h_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\propto_1}} \mathcal{L}[\mu b - \beta \frac{k_2 b}{k_4}]\right\} \tag{72}$$

Recall that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\propto_1}}\right\} = \frac{t^{\propto_1}}{\Gamma(\propto_1+1)} \tag{73}$$

$$h_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\propto_1}}\right\} \times \mathcal{L}^{-1}\mathcal{L}[\mu b - \beta \frac{k_2 b}{k_4}] \tag{74}$$

$$h_1 = \left(\mu b - \beta \frac{k_2 b}{k_4}\right) \frac{t^{\propto_1}}{\Gamma(\propto_1+1)} \tag{75}$$

From equation (52) we obtain p_1 to be,

$$p_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha_2}} \mathcal{L}[(pA) \frac{\partial^2 h_0}{\partial x^2} + \gamma h_0 b_0 - \zeta p_0]\right\} \tag{76}$$

$$p_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha_2}} \mathcal{L}[(pA) \frac{\partial^2 k_2 b/k_4}{\partial x^2} + \gamma k_2 b/k_4 b - \zeta(0)]\right\} \tag{77}$$

$$\frac{\partial^2 k_2 b/k_4}{\partial x^2} = 0 \quad \text{and} \quad \zeta(0) = 0 \tag{78}$$

$$p_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\right] \times \mathcal{L}^{-1} \mathcal{L}[\gamma \mathrm{b} k_2 b/k_4]. \tag{79}$$

Using the fact that

$$\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\right] = \frac{t^{\propto_2}}{\Gamma(\propto_2+1)} \tag{80}$$

and

$$\mathcal{L}^{-1} \mathcal{L}[\gamma \mathrm{b} k_2 b/k_4] = (\gamma \mathrm{b} k_2 b/k_4) \tag{81}$$

we have

$$p_1 = (\gamma \mathrm{b} k_2 b/k_4) \frac{t^{\propto_2}}{\Gamma(\propto_2+1)}. \tag{82}$$

From equation (53) we get

$$b_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha_3}} \mathcal{L}[Ap \frac{\partial h_0}{\partial x} + \varepsilon b_0(k - b_0)\right\} \tag{83}$$

which becomes

$$b_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\right] \times \mathcal{L}^{-1} \mathcal{L}\left[Ap \frac{\partial h_0}{\partial x} + \varepsilon b_0 \left(1 - \frac{b_0}{k}\right)\right] \tag{84}$$

and

$$b_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\right] \times \mathcal{L}^{-1} \mathcal{L}\left[Ap \frac{\partial k_2 b/k_4}{\partial x} + \varepsilon b_0 - \frac{\varepsilon b_0^2}{k}\right]. \tag{85}$$

Applying

$$Ap \frac{\partial k_2 b/k_4}{\partial x} = 0, \tag{86}$$

we obtain

$b_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\right] \times \mathcal{L}^{-1}\,\mathcal{L}\left[\varepsilon b_0 - \frac{\varepsilon b_0^2}{k}\right].$ (87)

We recall from 67 that

$M_n = \frac{1}{\Gamma(n+1)}\frac{d}{d\lambda^n}\left[\sum_{k=0}^{n}\lambda^k\, h_k \sum_{k=0}^{n}\lambda^k\, b_k\right]\Big|_{\lambda=0}$

$M_0 = \frac{1}{\Gamma(0+1)}\frac{d}{d\lambda^0}\left[\sum_{k=0}^{n}\lambda^0\, b_0 \sum_{k=0}^{n}\lambda^0\, b_0\right]\Big|_{\lambda=0}$ (88)

$M_0 = \frac{d}{d\lambda^0}(\lambda^0 b_0 + \lambda^1 h_1)(\lambda^0 b_0 + \lambda^1 b_1)$ (89)

Therefore

$M_0 = b_0 b_0$ (90)

Equation 87 becomes

$b_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\right] \times \mathcal{L}^{-1}\,\mathcal{L}\left[\varepsilon b_0 - \frac{\varepsilon M_0}{k}\right]$ (91)

But $\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\right] = \frac{t^{\propto_3}}{\Gamma(\propto_3+1)}$ (92)

Therefore

$b_1 = \left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3}}{\Gamma(\propto_3+1)}$ (93)

We consider equation 54 to obtain a_1

$a_1 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_4}}\mathcal{L}[Ap\frac{\partial^2 h_0}{\partial x^2} + \aleph a_0(k - a_0)]]$ (94)

$a_1 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_4}}\mathcal{L}[Ap\frac{\partial^2 k_2 b/k_4}{\partial x^2} + \aleph a_0(k - a_0)]]$ (95)

$Ap\frac{\partial^2 k_2 b/k_4}{\partial x^2} = 0$ (96)

$a_1 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_4}}\mathcal{L}[\,\aleph a_0(k - a_0)]]$ (97)

$a_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_4}}\right] \times \mathcal{L}^{-1}\mathcal{L}[\aleph a_0(k - a_0)]$ (98)

$a_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_4}}\right] \times \mathcal{L}^{-1}\mathcal{L}\left[\aleph a_0\left(1 - \frac{a_0}{k}\right)\right]$ (99)

$a_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_4}}\right] \times \mathcal{L}^{-1}\mathcal{L}\left[\aleph a_0 - \frac{\aleph a_0^2}{k}\right]$ (100)

But $\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_4}}\right] = \frac{t^{\propto_4}}{\Gamma(\propto_4+1)}$ (101)

$a_1 = \left[\aleph a_0 - \frac{\aleph a_0^2}{k}\right]\frac{t^{\propto_4}}{\Gamma(\propto_4+1)}$ (102)

$a_1 = \left[\aleph a - \frac{\aleph a^2}{k}\right]\frac{t^{\propto_4}}{\Gamma(\propto_4+1)}$ (103)

To obtain c_1 consider equation 55 as follows

$c_1 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_5}}\mathcal{L}[D_h\frac{\partial^2 h_0}{\partial x^2} + yb_0 - gc_0]]$ (104)

$c_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_5}}\right] \times \mathcal{L}^{-1}\mathcal{L}[D_h\frac{\partial^2 k_2 b/k_4}{\partial x^2} + yb_0 - g(0)]$ (105)

$\frac{\partial^2 k_2 b/k_4}{\partial x^2} = 0$ (106)

$c_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_5}}\right] \times \mathcal{L}^{-1}\mathcal{L}[yb_0 - g(0)]$ (107)

But $\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_5}}\right] = \frac{t^{\propto_5}}{\Gamma(\propto_5+1)}$ (108)

$$c_1 = \frac{t^{\propto_5}}{\Gamma(\propto_5+1)} \times yb \quad (109)$$

$$c_1 = (yb)\frac{t^{\propto_5}}{\Gamma(\propto_5+1)} \quad (110)$$

We obtain r_1 from equation 56 as follows

$$r_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_6}}\right]\mathcal{L}^{-1}\mathcal{L}[D_h\frac{\partial^2 h_0}{\partial x^2} + \sigma b_0 - \theta r_0] \quad (111)$$

$$r_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_6}}\right]\mathcal{L}^{-1}\mathcal{L}[D_h\frac{\partial^2 k_2 b/k_4}{\partial x^2} + \sigma b - \theta(0)] \quad (112)$$

$$D_h\frac{\partial^2 k_2 b/k_4}{\partial x^2} = 0 \quad (113)$$

$$r_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_6}}\right] \times \mathcal{L}^{-1}\mathcal{L}[\sigma b - \theta(0)] \quad (114)$$

$$r_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_6}}\right] \times \mathcal{L}^{-1}\mathcal{L}[\sigma b] \quad (115)$$

But $\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_6}}\right] = \frac{t^{\propto_6}}{\Gamma(\propto_6+1)}$ (116)

$$r_1 = [\sigma b]\frac{t^{\propto_6}}{\Gamma(\propto_6+1)} \quad (117)$$

To obtain h_2 we revert to equation 51

$$h_2 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_1}}\mathcal{L}[D_h\frac{\partial^2 h_1}{\partial x^2} + \mu b_1 - \beta h_1]] \quad (118)$$

$$h_2 = \mu\left[\varepsilon b_0 - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3+\propto_1}}{\Gamma(\propto_3+\propto_1+1)} - \beta\left[\mu b - \beta\frac{k_2 b}{k_4}\right]\frac{t^{2\propto_1}}{\Gamma(2\propto_1+1)} \quad (119)$$

To obtain p_2 we revert to equation 52

$$p_2 = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha_2}}\mathcal{L}\left[\frac{\partial}{\partial x}(pA)\frac{\partial h_1}{\partial x} + \gamma h_1 b_1 - \zeta p_1\right]\right\} \quad (120)$$

$$p_2 = \gamma\varepsilon\mu b^2\frac{t^{\propto_1+\alpha_2+\propto_3}}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} - \gamma\varepsilon\frac{b^3}{k}\frac{t^{\propto_1+\alpha_2+\propto_3}}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} - \gamma\varepsilon\beta b\frac{k_2 b}{k_4}\frac{t^{\propto_1+\alpha_2+\propto_3}}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} + \gamma\varepsilon\beta b^2\frac{k_2 b}{k.k_4}\frac{t^{\propto_1+\alpha_2+\propto_3}}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} - \zeta(\gamma b k_2 b/k_4)\frac{t^{2\propto_2}}{\Gamma(2\propto_2+1)} \quad (121)$$

To obtain b_2, we revert to equation 53

$$b_2 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\left[Ap\frac{\partial h_1}{\partial x} + \varepsilon b_1(k - b_1)\right]\right] \quad (122)$$

$$b_2 = \left[\varepsilon b_1 - \frac{\varepsilon b_1^2}{k}\right]\frac{t^{\propto_3}}{\Gamma(\propto_3+1)} \quad (123)$$

$$b_2 = \left[\varepsilon b_1 - \frac{\varepsilon j_1}{k}\right]\frac{t^{\propto_3}}{\Gamma(\propto_3+1)} \quad (124)$$

$$J_n = \frac{1}{\Gamma(n+1)}\frac{d^n}{d\lambda^n}[\sum_{k=0}^{n}\lambda^k h_k \sum_{k=0}^{n}\lambda^k h_k]|_{\lambda=0} \quad (125)$$

$$J_n = \frac{1}{\Gamma(1+1)}\frac{d}{d\lambda^1}[(\lambda^0 b_0 + \lambda^1 b_1)(\lambda^0 b_0 + \lambda^1 b_1)]\,|_{\lambda=0} \quad (126)$$

$$J_1 = \frac{1}{\Gamma(1+1)}\frac{d^1}{d\lambda^1}[(b_0 + \lambda^1 b_1)(b_0 + \lambda^1 b_1)]\,|_{\lambda=0} \quad (127)$$

$$J_1 = \frac{1}{\Gamma(1+1)}\frac{d^1}{d\lambda^1}(b_0^2 + \lambda^1 b_0 b_1 + \lambda^1 b_0 b_1 + \lambda^2 b_1^2) \quad (128)$$

$$J_1 = 2b_0 b_1 \quad (129)$$

$$J_1 = 2b\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3}}{\Gamma(\propto_3+1)} \quad (130)$$

Substituting the values of J_1 and b_1 into equation (123)

$$b_2 = \varepsilon\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{2\propto_3}}{\Gamma(2\propto_3+1)} - \left[\frac{\varepsilon 2b(\varepsilon b - \frac{\varepsilon b^2}{k})\frac{t^{2\propto_3}}{\Gamma(2\propto_3+1)}}{k}\right] \quad (131)$$

We obtain a_2 from equation (54) as follows

$$a_2 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_4}}\mathcal{L}[Ap\frac{\partial^2 h_1}{\partial x^2} + \aleph a_1(k - a_1)]] \quad (132)$$

$$a_2 = \aleph\left(\aleph a - \frac{\aleph a^2}{k}\right)\left[\frac{k_2 b}{k_4}\right]\frac{t^{2\propto_4}}{\Gamma(2\propto_4+1)} - \frac{\aleph\left(\aleph a - \frac{\aleph a^2}{k}\right)\left[\frac{k_2 b}{k_4}\right]\frac{t^{2\propto_4}}{\Gamma(2\propto_4+1)}}{k} \quad (133)$$

We obtain c_2 from (55) as follows

$$c_2 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_5}}\mathcal{L}[D_h\frac{\partial^2 h_1}{\partial x^2} + yb_1 - gc_1]] \quad (134)$$

$$c_2 = \left[y\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3+\propto_5}}{\Gamma(\propto_3+\propto_5+1)} - g(yb)\frac{t^{2\propto_5}}{\Gamma(2\propto_5+1)}\right] \quad (135)$$

We obtain r_2 from equation 56 as follows

$$r_2 = \mathcal{L}^{-1}[\frac{1}{s^{\alpha_6}}\ \mathcal{L}[D_h\frac{\partial^2 h_1}{\partial x^2} + \sigma b_1 - \theta r_1]]$$

$$r_2 = [\sigma\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3+\propto_6}}{\Gamma(\propto_3+\propto_6+1)} - \theta[\sigma b]\frac{t^{2\propto_6}}{\Gamma(2\propto_6+1)} \quad (136)$$

So, the equations for the model are given as

$$h(t) = \frac{k_2 b}{k_4} + \left(\mu b - \beta\frac{k_2 b}{k_4}\right)\frac{t^{\propto_1}}{\Gamma(\propto_1+1)} + \mu\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3+\propto_1}}{\Gamma(\propto_3+\propto_1+1)} - \beta\left[\mu b - \beta\frac{k_2 b}{k_4}\right]\frac{t^{2\propto_1}}{\Gamma(2\propto_1+1)} \quad (137)$$

$$p(t) = \left(\frac{\text{ɣb}k_2 b}{k_4}\right)\frac{t^{\propto_2}}{\Gamma(\propto_2+1)} + \text{ɣ}\varepsilon\mu b^2\frac{t^{\propto_1+\alpha_2}+\propto_3}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} - \text{ɣ}\varepsilon\frac{b^3}{k}\frac{t^{\propto_1+\alpha_2}+\propto_3}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} - \text{ɣ}\varepsilon\beta b\frac{k_2 b}{k_4}\frac{t^{\propto_1+\alpha_2+\propto_3}}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} + \text{ɣ}\varepsilon\beta b^2\frac{k_2 b}{k.k_4}\frac{t^{\propto_1+\alpha_2+\propto_3}}{\Gamma(\propto_1+\alpha_2+\propto_3+1)} - \zeta(\text{ɣb}k_2 b/k_4)\frac{t^{2\propto_2}}{\Gamma(2\propto_2+1)} \quad (138)$$

$$b(t) = b + \left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3}}{\Gamma(\propto_3+1)} + \varepsilon\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{2\propto_3}}{\Gamma(2\propto_3+1)} - \left[\frac{\varepsilon 2b(\varepsilon b - \frac{\varepsilon b^2}{k})\frac{t^{2\propto_3}}{\Gamma(2\propto_3+1)}}{k}\right] \quad (139)$$

$$a(t) = a + \left[\aleph a - \frac{\aleph a^2}{k}\right]\frac{t^{\propto_4}}{\Gamma(\propto_4+1)} + \aleph\left(\aleph a - \frac{\aleph a^2}{k}\right)\left[\frac{k_2 b}{k_4}\right]\frac{t^{2\propto_4}}{\Gamma(2\propto_4+1)} - \frac{\aleph\left(\aleph a - \frac{\aleph a^2}{k}\right)\left[\frac{k_2 b}{k_4}\right]\frac{t^{2\propto_4}}{\Gamma(2\propto_4+1)}}{k} \quad (140)$$

$$c(t) = (yb)\frac{t^{\propto_5}}{\Gamma(\propto_5+1)} + \left[y\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3+\propto_5}}{\Gamma(\propto_3+\propto_5+1)} - g(yb)\frac{t^{2\propto_5}}{\Gamma(2\propto_5+1)}\right] \quad (141)$$

$$r(t) = [\sigma b]\frac{t^{\propto_6}}{\Gamma(\propto_6+1)} + [\sigma\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{t^{\propto_3+\propto_6}}{\Gamma(\propto_3+\propto_6+1)} - \theta[\sigma b]\frac{t^{2\propto_6}}{\Gamma(2\propto_6+1)} \quad (142)$$

The general solution to the model is given as

$$u = \Omega\left[\frac{k_2 b}{k_4}\right]\left[\frac{1}{\xi}\right] + \frac{\Omega\left[\mu b - \beta\frac{k_2 b}{k_4}\right]\frac{1}{\Gamma(\propto_1+1)}}{e^{\xi t}}\left[\frac{t^{\propto_1+1}}{\propto_1+1} + \frac{\xi t^{\propto_1+2}}{\propto_1+2} + \frac{\xi^2 t^{\propto_1+3}}{2(\propto_1+3)}\right] + \frac{\Omega(\mu)\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{1}{\Gamma(\propto_3+\propto_1+1)}}{e^{\xi t}}\left[\frac{t^{\propto_3+\propto_1+1}}{\propto_3+\propto_1+1} + \frac{\xi t^{\propto_3+\propto_1+2}}{\propto_3+\propto_1+2} + \frac{\xi^2 t^{\propto_3+\propto_1+3}}{2(\propto_3+\propto_1+3)}\right] - \frac{\Omega(\beta)\left[\mu b - \beta\frac{k_2 b}{k_4}\right]\frac{1}{\Gamma(2\propto_1+1)}}{e^{\xi t}}\left[\frac{t^{2\propto_1+1}}{2\propto_1+1} + \frac{\xi t^{2\propto_1+2}}{2\propto_1+2} + \frac{\xi^2 t^{2\propto_1+3}}{2(2\propto_1+3)}\right] + \frac{\varphi\left(\frac{\text{ɣb}k_2 b}{k_4}\right)\frac{1}{\Gamma(\propto_2+1)}}{e^{\xi t}}\left[\frac{t^{\propto_2+1}}{\propto_2+1} + \frac{\xi t^{\propto_2+2}}{\propto_2+2} + \frac{\xi^2 t^{\propto_2+3}}{2(\propto_2+3)}\right] + \frac{\varphi(\varepsilon\text{ɣ}\mu b^2)\frac{1}{\Gamma(\propto_1+\alpha_2+\alpha_3+1)}}{e^{\xi t}}\left[\frac{t^{\propto_1+\propto_2+\propto_3+1}}{\propto_1+\propto_2+\propto_3+1} + \frac{\xi t^{\propto_1+\propto_2+\propto_3+2}}{\propto_1+\propto_2+\propto_3+2} + \right.$$

$$\frac{\xi^2 t^{\alpha_1+\alpha_2+\alpha_3+3}}{2(\alpha_1+\alpha_2+\alpha_3+3)}\Bigg] + \frac{\varphi\left(\varepsilon\gamma\frac{b^3}{k}\right)\frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}}{e^{\xi t}}\left[\frac{t^{\alpha_1+\alpha_2+\alpha_3+1}}{\alpha_1+\alpha_2+\alpha_3+1} + \frac{\xi t^{\alpha_1+\alpha_2+\alpha_3+2}}{\alpha_1+\alpha_2+\alpha_3+2} + \frac{\xi^2 t^{\alpha_1+\alpha_2+\alpha_3+3}}{2(\alpha_1+\alpha_2+\alpha_3+3)}\right] -$$
$$\frac{\varphi\left(\varepsilon\gamma\beta b\frac{k_2 b}{k_4}\right)\frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}}{e^{\xi t}}\left[\frac{t^{\alpha_1+\alpha_2+\alpha_3+1}}{\alpha_1+\alpha_2+\alpha_3+1} + \frac{\xi t^{\alpha_1+\alpha_2+\alpha_3+2}}{\alpha_1+\alpha_2+\alpha_3+2} + \frac{\xi^2 t^{\alpha_1+\alpha_2+\alpha_3+3}}{2(\alpha_1+\alpha_2+\alpha_3+3)}\right] +$$
$$\frac{\varphi\left(\varepsilon\gamma\beta b^2\frac{k_2 b}{k.k_4}\right)\frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}}{e^{\xi t}}\left[\frac{t^{\alpha_1+\alpha_2+\alpha_3+1}}{\alpha_1+\alpha_2+\alpha_3+1} + \frac{\xi t^{\alpha_1+\alpha_2+\alpha_3+2}}{\alpha_1+\alpha_2+\alpha_3+2} + \frac{\xi^2 t^{\alpha_1+\alpha_2+\alpha_3+3}}{2(\alpha_1+\alpha_2+\alpha_3+3)}\right] - \frac{\varphi\zeta\left(\frac{\gamma b k_2 b}{k_4}\right)\frac{1}{\Gamma(2\alpha_2+1)}}{e^{\xi t}}\Bigg[\frac{t^{2\alpha_2+1}}{2\alpha_2+1} +$$
$$\frac{\xi t^{2\alpha_2+2}}{2\alpha_2+2} + \frac{\xi^2 t^{2\alpha_2+3}}{2(2\alpha_2+3)}\Bigg] + dbe^{\xi t}\left[\frac{1}{\xi}\right] - \frac{d\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{1}{\Gamma(\alpha_3+1)}}{e^{\xi t}}\left[\frac{t^{\alpha_3+1}}{\alpha_3+1} + \frac{\xi t^{\alpha_3+2}}{\alpha_3+2} + \frac{\xi^2 t^{\alpha_3+3}}{2(\alpha_3+3)}\right] +$$
$$\frac{d\varepsilon\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{1}{\Gamma(2\alpha_3+1)}}{e^{\xi t}}\left[\frac{t^{2\alpha_3+1}}{2\alpha_3+1} + \frac{\xi t^{2\alpha_3+2}}{2\alpha_3+2} + \frac{\xi^2 t^{2\alpha_3+3}}{2(2\alpha_3+3)}\right] + \frac{d\left[\frac{\varepsilon 2b\left(\varepsilon b - \frac{\varepsilon b^2}{k}\right)\frac{1}{\Gamma(2\alpha_3+1)}}{k}\right]}{e^{\xi t}}\Bigg[\frac{t^{2\alpha_3+1}}{2\alpha_3+1} + \frac{\xi t^{2\alpha_3+2}}{2\alpha_3+2} +$$
$$\frac{\xi^2 t^{2\alpha_3+3}}{2(2\alpha_3+3)}\Bigg] + \tau a e^{\xi t}\left[\frac{e^{\xi}}{\xi}\right] + \frac{\tau\left[\aleph a - \frac{\aleph a^2}{k}\right]\frac{1}{\Gamma(\alpha_4+1)}}{e^{\xi t}}\left[\frac{t^{\alpha_4+1}}{\alpha_4+1} + \frac{\xi t^{\alpha_4+2}}{\alpha_4+2} + \frac{\xi^2 t^{\alpha_4+3}}{2(\alpha_4+3)}\right] +$$
$$\frac{\tau\aleph\left(\aleph a - \frac{\aleph a^2}{k}\right)\left[\frac{k_2 b}{k_4}\right]\frac{1}{\Gamma(2\alpha_4+1)}}{e^{\xi t}}\left[\frac{t^{2\alpha_4+1}}{2\alpha_4+1} + \frac{\xi t^{2\alpha_4+2}}{2\alpha_4+2}\frac{\xi^2 t^{2\alpha_4+3}}{2(2\alpha_4+3)}\right] + \frac{\tau\frac{\aleph\left(\aleph a - \frac{\aleph a^2}{k}\right)\left[\frac{k_2 b}{k_4}\right]\frac{1}{\Gamma(2\alpha_4+1)}}{k}}{e^{\xi t}}\Bigg[\frac{t^{2\alpha_4+1}}{2\alpha_4+1} + \frac{\xi t^{2\alpha_4+2}}{2\alpha_4+2} +$$
$$\frac{\xi^2 t^{2\alpha_4+3}}{2(2\alpha_4+3)}\Bigg] + \frac{\eta(yb)\frac{1}{\Gamma(\alpha_5+1)}=}{e^{\xi t}}\left[\frac{t^{\alpha_5+1}}{\alpha_5+1} + \frac{\xi t^{\alpha_5+2}}{\alpha_5+2} + \frac{\xi^2 t^{\alpha_5+3}}{2(\alpha_5+3)}\right] + \frac{y\eta\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\frac{1}{\Gamma(\alpha_3+\alpha_5+1)}}{e^{\xi t}}\Bigg[\frac{t^{\alpha_3+\alpha_5+1}}{\alpha_3+\alpha_5+1} + \frac{\xi t^{\alpha_3+\alpha_5+2}}{\alpha_3+\alpha_5+2} +$$
$$\frac{\xi^2 t^{\alpha_3+\alpha_5+3}}{2(\alpha_3+\alpha_5+3)}\Bigg] - \frac{\eta y g(yb)\frac{1}{\Gamma(2\alpha_5+1)}}{e^{\xi t}}\left[\frac{t^{2\alpha_5+1}}{2\alpha_5+1} + \frac{\xi t^{2\alpha_5+2}}{2\alpha_5+2} + \frac{\xi^2 t^{2\alpha_5+3}}{2(2\alpha_5+3)}\right] + \frac{v[\sigma b]\frac{1}{\Gamma(\alpha_6+1)}}{e^{\xi t}}\Bigg[\frac{t^{\alpha_6+1}}{\alpha_6+1} + \frac{\xi t^{\alpha_6+2}}{\alpha_6+2} +$$
$$\frac{\xi^2 t^{\alpha_6+3}}{2(\alpha_6+3)}\Bigg] + \frac{v\left[\sigma\left[\varepsilon b - \frac{\varepsilon b^2}{k}\right]\right]\frac{1}{\Gamma(\alpha_3+\alpha_6+1)}}{e^{\xi t}}\left[\frac{t^{\alpha_3+\alpha_6+1}}{\alpha_3+\alpha_6+1} + \frac{\xi t^{\alpha_3+\alpha_6+2}}{\alpha_3+\alpha_6+2} + \frac{\xi^2 t^{\alpha_3+\alpha_6+3}}{2(\alpha_3+\alpha_6+3)}\right] - \frac{v\theta[\sigma b]\frac{1}{\Gamma(2\alpha_6+1)}}{e^{\xi t}}\Bigg[\frac{t^{2\alpha_6+1}}{2\alpha_6+1} +$$
$$\frac{\xi t^{2\alpha_6+2}}{2\alpha_6+2} + \frac{\xi^2 t^{2\alpha_6+3}}{2(2\alpha_6+3)}\Bigg] \qquad (143)$$

IV. DISCUSSION, SUMMARY AND CONCLUSION

4.1 Assumptions

i. We assumed a rehabilitation process that starts at the point $x = 0$ and center at $x = L$ with the distance from the starting point given as $x = w$.

ii. We assumed six rehabilitation factors: liquidity restoration, price momentum recovery, market sentiment recovery, volatility reduction, price recovery and price stabilization.

From the existing mathematical model, the parameters were derived from experimental literature. The fixed parameters are given as follows:

$$k_2 = 10, \zeta = 20\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0.5,\ \lambda = 10, k_4 = 0.1, g = 50$$
$$D_h = 10$$

Estimated parameters are as follows: $\mu = 5$, $\beta = 0.5$, $\varepsilon = 0.1$.

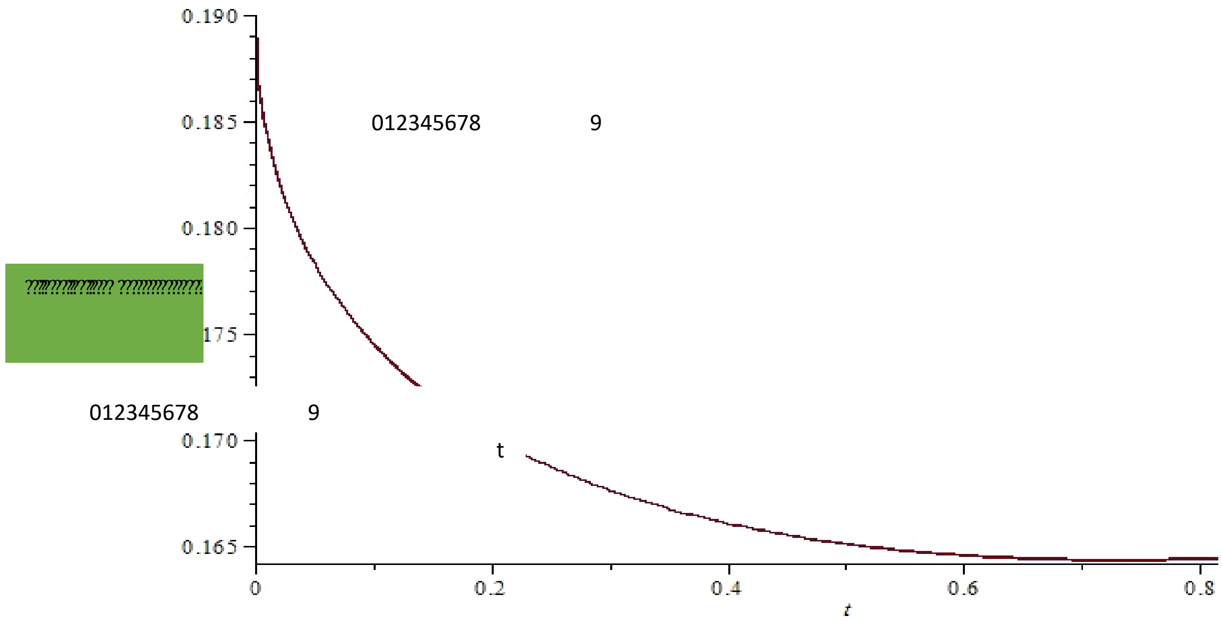

Fig.1 shows the liquidity restoration for the first 10 weeks after the stock market changed.

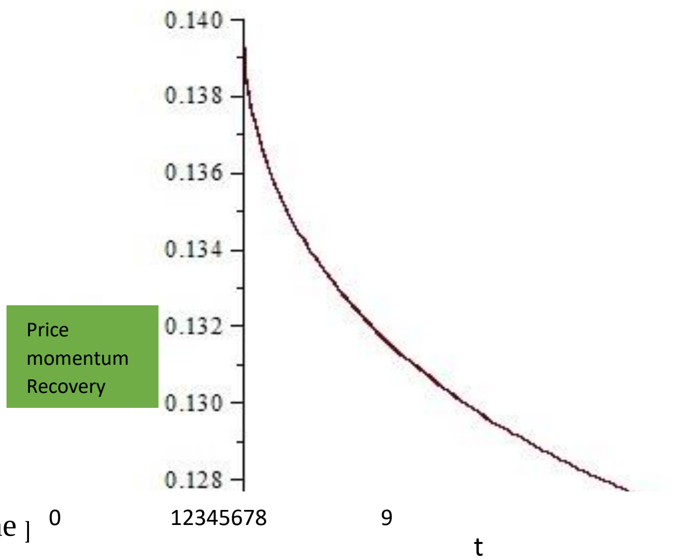

Fig 2 shows the

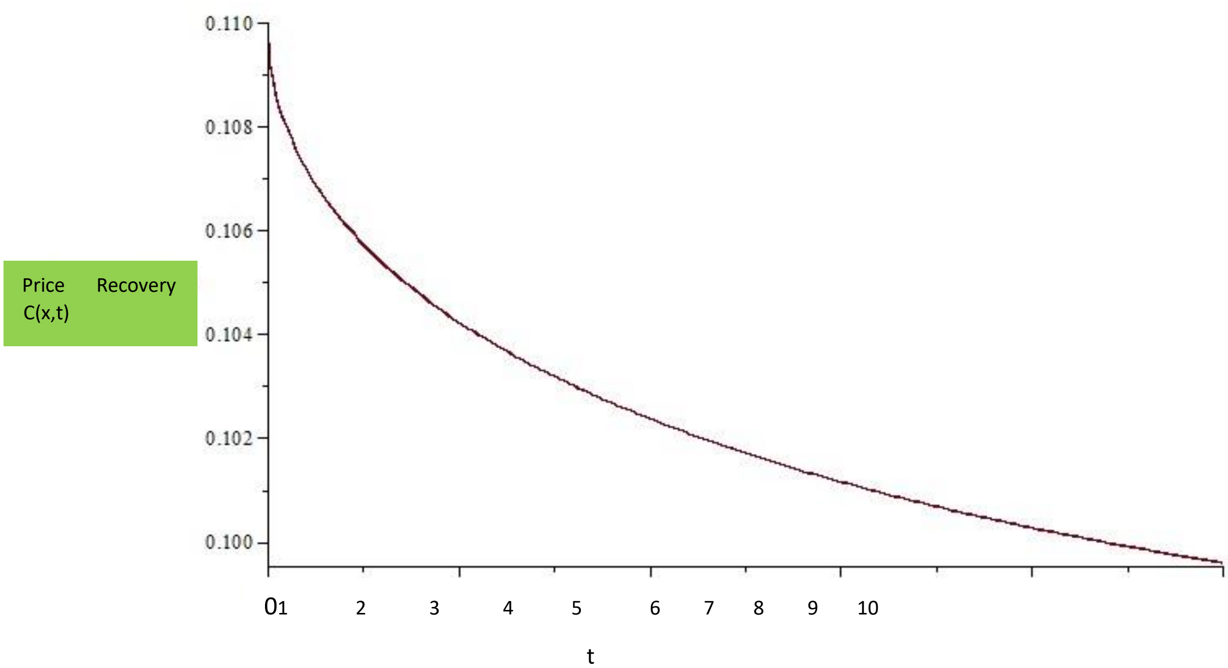

Fig. 5. shows the

0 12345678910

Price
Stabilizatio
r(t)

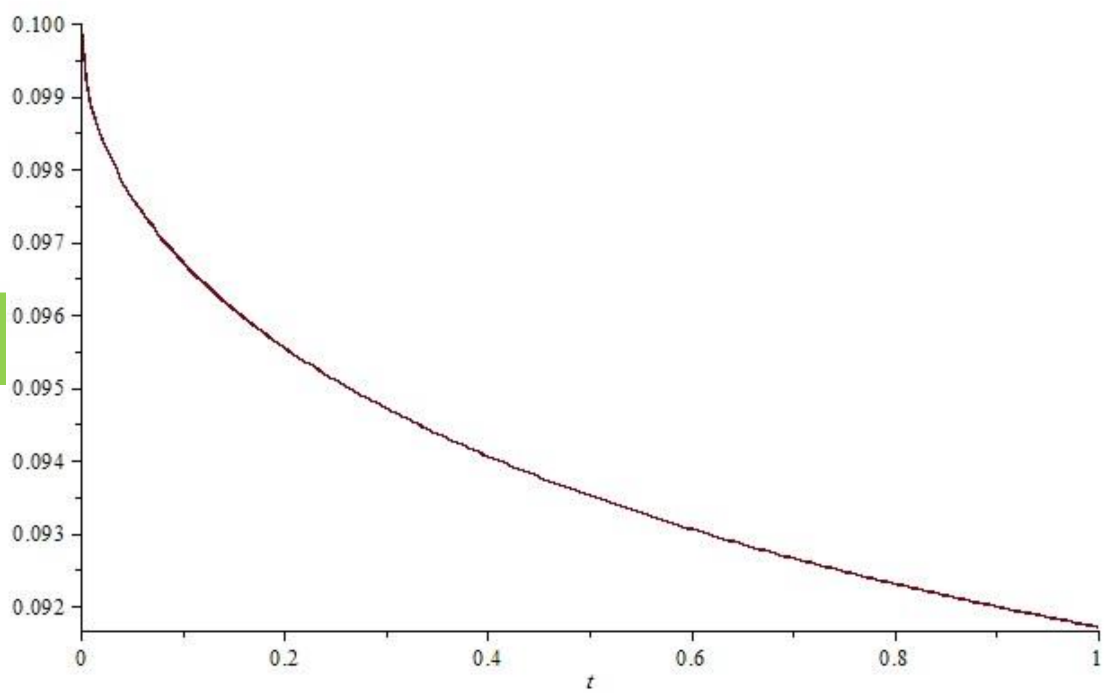

Fig. 6 shows the Price Stabilization for the first 10 weeks after the stock market challenged

We used the solution obtained from the model to plot graphs of liquidity restoration, price momentum recovery, market sentiment recovery, volatility reduction, price recovery and price stabilization against time respectively. We then tried to see the contribution of each of these factors to the rehabilitation process of the stock market. We can also see from the graphs that as the time increases, the rehabilitation takes place, and the impact the six factors input helps to return the stock market to the normal level of performance.

4.2 Summary

This paper introduces a sophisticated mathematical framework that employs fractional calculus to model the rehabilitation dynamics of stock markets. The focus is on how markets recover after experiencing significant disruptions, such as financial crises, using the nuanced and memory-sensitive tools provided by fractional calculus. Traditional models often fail to capture the

intricacies of market recovery, particularly the intertwined effects of liquidity, sentiment, and volatility. This study leverages the unique properties of fractional derivatives and integrals to address these challenges.
The model developed in this paper addresses the following key aspects of stock market rehabilitation:

1. **Liquidity Restoration**: The model incorporates fractional-order derivatives to capture the gradual restoration of liquidity in the market. The non-local nature of fractional calculus allows the model to consider the influence of past market states on the current liquidity conditions, reflecting the time-dependent process of re-establishing normal trading volumes.
2. **Price Momentum Recovery**: By modeling price momentum recovery with fractional derivatives, the paper accounts for the persistence and memory effects that influence price movements over time. This approach provides a more accurate representation of how price momentum builds up during the market's recovery phase.
3. **Market Sentiment Recovery**: The recovery of market sentiment is modeled using fractional integrals, which allow the model to capture the impact of past sentiment on current market conditions. This aspect of the model reflects how investor confidence gradually rebuilds, influenced by a combination of historical performance and emerging market signals.
4. **Volatility Reduction**: The model addresses volatility reduction as a key component of market rehabilitation. Fractional calculus is used to model the decay of volatility over time, providing a more nuanced understanding of how volatility diminishes as the market stabilizes.
5. **Price Recovery**: The process of price recovery is modeled with fractional differential equations that capture the time-dependent nature of price adjustments. This allows the model to reflect the gradual convergence of prices towards their intrinsic values as market conditions improve.
6. **Price Stabilization**: Finally, the model includes a mechanism for price stabilization, where fractional calculus is used to describe the smoothing effects that occur as the market approaches a new equilibrium. This aspect of the model highlights the role of long-term memory in maintaining price stability after a period of recovery.

The paper presents a detailed mathematical formulation of the proposed fractional calculus model, supported by theoretical analysis and empirical validation. Simulations are conducted to demonstrate the model's ability to replicate real-world market recovery scenarios, showing that it can effectively capture the complex, interrelated dynamics of market rehabilitation.

4.3Conclusion

This paper has introduced a novel framework for understanding and modeling the rehabilitation dynamics of stock markets through the application of fractional calculus. By addressing the multifaceted processes involved in market recovery—such as liquidity restoration, price momentum recovery, market sentiment recovery, volatility reduction, price recovery, and price stabilization—the study has provided a comprehensive tool that captures the complexity of financial systems in the aftermath of significant disruptions.

The application of fractional calculus has proven to be particularly effective in this context due to its ability to model systems with memory and hereditary properties, which are inherent in financial markets. Traditional models, which often rely on integer-order derivatives, fall short in accounting for the long-term dependencies and interrelated dynamics that characterize market behavior during recovery. In contrast, fractional calculus, with its capacity to incorporate the effects of past states on present dynamics, offers a more accurate and realistic depiction of market rehabilitation.

The fractional differential equations formulated in this study have demonstrated their utility in capturing the gradual processes of recovery. Liquidity restoration and price momentum recovery, influenced by historical market conditions, have been effectively modeled through fractional derivatives that account for the slow and persistent nature of these processes. Similarly, market sentiment recovery and volatility reduction, which are critical to the stabilization of markets, have been shown to depend on past market behaviors, justifying the use of fractional integrals to represent these effects.

Furthermore, the model's ability to predict price recovery and stabilization provides a valuable insight into the mechanisms that drive markets back to equilibrium. By incorporating long-term memory effects, the model reflects the smoothing and stabilizing forces that emerge as markets recover, offering a more precise tool for forecasting the trajectory of prices post-disruption.

In conclusion, this paper contributes significantly to the field of financial mathematics by introducing a fractional calculus-based approach to modeling stock market rehabilitation. The proposed model not only advances our theoretical understanding of market recovery processes but also offers practical implications for financial analysts, policymakers, and traders seeking to navigate periods of market instability. Future research could extend this framework to explore its applicability to other financial phenomena, further establishing fractional calculus as a powerful tool in financial modeling.

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